Pre-class Warm-up!!!

Let $f(x,y) = 3x^2 - 6xy + 2y^3$. Three questions:

= Taylor approximation of order 2

1. What is the Taylor polynomial of degree 2 for f at the point $(x,y) = (0,0)$?

a. $3x^2 - 6xy + 2y^3$.

✓ b. $3x^2 - 6xy$

c. $3x^2 - 6xy + 2y^2$

Two approaches

1. b. is the best quadratic approximation to f.

2. It is

$$F(0,0) + \frac{\partial f}{\partial x}\Big|_{(0,0)} x + \frac{\partial f}{\partial y}\Big|_{(0,0)} y + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}\Big|_{(0,0)} x^2 + \frac{\partial^2 f}{\partial y \partial x}\Big|_{(0,0)} xy + \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}\Big|_{(0,0)} y^2$$

2. What is the Hessian matrix for f at the point $(x,y) = (0,0)$? d.

a. $\begin{bmatrix} 3 & -6 \\ -6 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 6 & -6 \\ -6 & 12 \end{bmatrix}$

c. $\begin{bmatrix} 3 & -6 \\ -6 & 0 \end{bmatrix}$

✓ d. $\begin{bmatrix} 6 & -6 \\ -6 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 6 & -6 \\ -6 & 4 \end{bmatrix}$

3. Fact: the Taylor polynomial of degree 2 for f at the point $(x,y) = (1,1)$ is

$-1 + 3(x-1)^2 - 6(x-1)(y-1) + 6(y-1)^2$

What is the Hessian matrix for f at $(x,y) = (1,1)$? b.

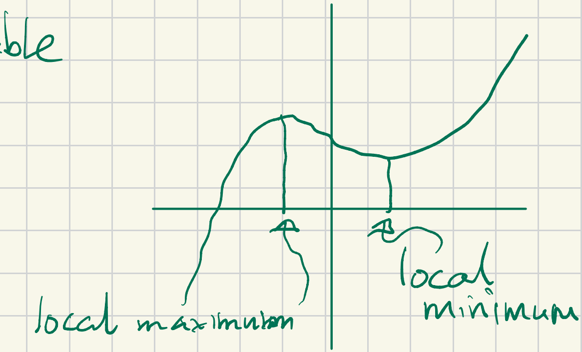
Recall the Hessian matrix:

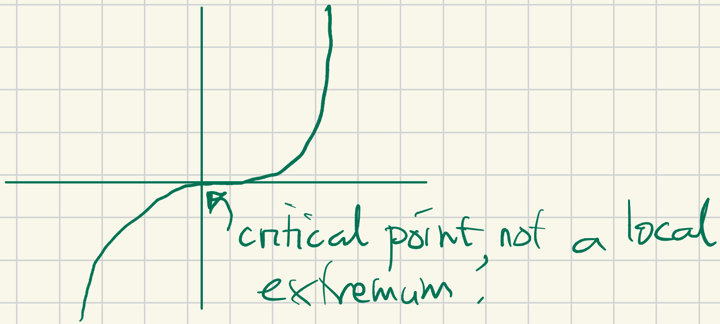
$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \quad \text{at } (x_0, y_0)$$

Section 3.3: Extrema of real valued functions
First half: local extrema

We learn
- what do we mean by a local maximum or a local minimum?
- The term: critical point
- How do we find local extrema?
- Just like the 1-variable case, there is a 2nd derivative test, and there are more possibilities than just having a local maximum or minimum: saddle points.

1-variable





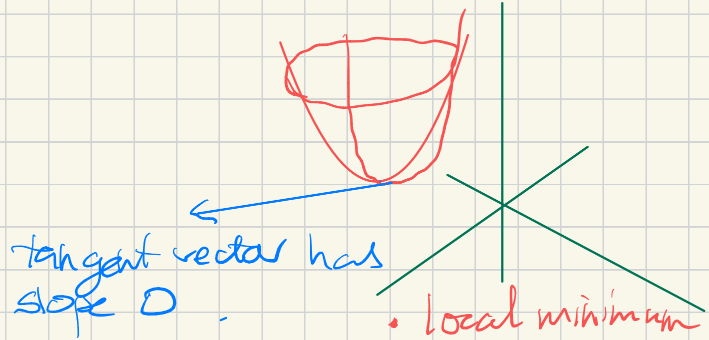
Definitions:

Let f : R^n -> R, let a be a vector in R^n.

f has a local minimum at a if there is a neighborhood of a so that f(a) is no bigger than any other values of f on this neighborhood.

f has a local maximum at a if same ... f is at least as big as f on all points in the neighborhood.

f has an local extremum at a means f has a local maximum or minimum at a.



How do we find local extrema?

Definition: a is a critical point means $\left.\frac{\partial f}{\partial x_i}\right|_a = 0$ for every i.

Local extremum => critical point.

Example. Let f(x,y) = 3x^2 - 6xy + 2y^3. Find all the critical points of f (and whether they are local maxima, minima, or saddle points).

Solution. We solve $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$.

Notation $f_x = \frac{\partial f}{\partial x}$ $\quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$

$\frac{\partial f}{\partial x} = 6x - 6y = 0$, $\frac{\partial f}{\partial y} = -6x + 6y^2 = 0$

Equation 1 $y = x$ $\quad$ Eqn 2 : $x = y^2$

$y = y^2$, $y^2 - y = y(y-1) = 0$, $y = 0$ or 1

$(x,y) = (0,0)$ or $(1,1)$.

These are the critical points.

Find the Taylor polynomial of degree 2 of f(x,y) = 3x^2 - 6xy + 2y^3 about (x,y) = (0,0).

Solution. $f_{xx} = 6$ $\quad f_{xy} = -6$ $\quad f_{yy} = 12y$.

At $(x,y) = (0,0)$ these are $6, -6, 0$.

Taylor poly of degree 2 is

$$f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2!}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}f_{yy}(0,0)$$

$= 3x^2 - 6xy$

The Hessian matrix is

$$H = \begin{bmatrix} 6 & -6 \\ -6 & 0 \end{bmatrix}$$

Notice that 3x^2 - 6xy = 3[(x-y)^2 - y^2] is a saddle point. On $y = 0$ it is ≥ 0

On $x = y$ it ≤ 0

Find the Taylor polynomial of degree 2 of
f(x,y) = 3x^2 - 6xy + 2y^3 about (x,y) = (1,1).

The Hessian matrix is

$$H = \begin{bmatrix} 6 & -6 \\ -6 & 12 \end{bmatrix}$$

Write $u = (x-1)$ $v = (y-1)$

Notice that

3u^2 - 6uv + 6v^2

= 3[(u-v)^2 + v^2], which has a minimum when (u,v) = (0,0). In every direction the function gets bigger

The second derivative test for local extrema.

If a is a critical point of f then it is either a local minimum, a local maximum, or a saddle point.

Sufficient criterion for a local minimum:

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_a > 0 \quad \text{and } \det H > 0$$

Sufficient criterion for a local maximum:

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_a < 0 \quad \text{and } \det H > 0$$

Sufficient criterion for a saddle point:

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_a \neq 0 \quad \text{and } \det H < 0$$

In other cases, the test is inconclusive, we can't tell. This happens when

Let's try this out with our Hessian matrices:

$$H = \begin{bmatrix} 6 & -6 \\ -6 & 0 \end{bmatrix} \qquad \det H = -36 < 0$$

saddle point.

$$H = \begin{bmatrix} 6 & -6 \\ -6 & 12 \end{bmatrix} \qquad \det H = 72 - 36 = 36 > 0$$

$$\left. f_{xx} \right|_{(1,1)} = 6 > 0$$

local minimum.

Pre-class Warm-up!!!

Do you remember the criterion on the Hessian matrix for a local minimum / local maximum / saddle point?

If the Hessian matrix is

$$H = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$ saddle point.

do we get a

a. Local minimum

b. Local maximum

c. Saddle point

d. Can't tell

What about the matrices:

$$H = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \qquad H = \begin{bmatrix} -3 & 2 \\ 2 & -6 \end{bmatrix}$$

minimum $\det H > 0$, $f_{xx} = 3 > 0$

$$H = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

saddle point $\det H < 0$ $f_{xx} \neq 0$

$$H = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

maximum $\det H > 0$ $f_{xx} = -3 < 0$

$$H = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$$

$f(x,y) = \frac{3}{2}x^2 + \frac{2}{2}y^2$ has $H = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ minimum

$f(x,y) = \frac{3}{2}x^2 - \frac{2}{2}y^2$ has $H = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ saddle point.

$f(x,y) = \frac{-3}{2}x^2 - \frac{2}{2}y^2$ has $H = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$ maximum

We did a test for local extrema when there are two variables, and there is a more general test for more variables.

Global maxima and minima

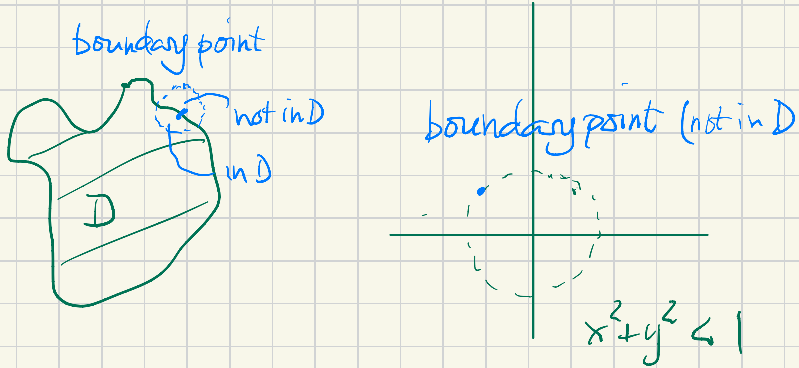
We learn

- on a closed bounded region of R^n a continuous function always has a point where it takes a maximum value and a point where it takes a minimum value. (There may be more than one such point.)
- What do bounded and closed mean? What is the boundary? What is the interior?
- How to find global extrema.

Definitions of boundary, closed, open

For a set D contained in $\mathbb{R}^n$, v in $\mathbb{R}^n$ is a boundary point of D if every solid ball center v of arbitrarily small radius contains a point of D and a point not in D.

D is closed if it contains all its boundary points.



To find global extrema: f is a function defined on a closed region D.

Step 1. Find the local extrema on the interior of $D = D - \partial D$

Step 2 Find the extrema of f on the boundary. Do this by parametrizing ∂D. (In section 3.4 the method of Lagrange multipliers also does.)

Step 3 Compare the values of f from Steps 1 & 2.

Step 3. We compare the values of f at

$$\left(\tfrac{1}{2}, \tfrac{1}{2}\right), \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \left(-\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right)$$

f is $\frac{1}{2}$ (minimum), $2-\sqrt{2}$, $2+\sqrt{2}$ (maximum).

Example: Find the maximum and minimum values of

f(x,y) = x^2 + y^2 - x - y + 1

on the unit disk x^2 + y^2 ≤ 1.

Solution: Step 1 Find the local extrema of f on the interior $x^2+y^2<1$

Critical points: $\frac{\partial f}{\partial x} = 2x-1 = 0$, $x = \frac{1}{2}$

$\frac{\partial f}{\partial y} = 2y-1 = 0$ $y = \frac{1}{2}$. One critical point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

If we wish, test if this is max or min.

Step 2 Parametrize the boundary

$c(t) = (\cos(t), \sin(t))$. Find the maxima & minima of $f(c(t)) = 1 - \cos(t) - \sin(t) + 1$

$\frac{d}{dt} f(c(t)) = \sin(t) - \cos(t) = 0$

$\sin(t) = \cos(t)$, $t = \frac{\pi}{4}$ or $\frac{5\pi}{4}$